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Nonclassical truth with classical strength.
A proof-theoretic analysis of compositional truth over HYPE.

Martin Fischer, Carlo Nicolai, and Pablo Dopico Fernandez

Abstract. Questions concerning the proof-theoretic strength of classical versus non-classical theories of truth have received some attention recently. A particularly convenient case study concerns classical and nonclassical axiomatizations of fixed-point semantics. It is known that nonclassical axiomatizations in four- or three-valued logics are substantially weaker than their classical counterparts. In this paper we consider the addition of a suitable conditional to First-Degree Entailment – a logic recently studied by Hannes Leitgeb under the label 'HYPE'. We show in particular that, by formulating the theory PKF over HYPE, one obtains a theory that is sound with respect to fixed-point models, while being proof-theoretically on a par with its classical counterpart KF. Moreover, we establish that also its schematic extension – in the sense of Feferman – is as strong as the schematic extension of KF, thus matching the strength of predicative analysis.

1. Introduction

The question whether there are non-classical formal systems of primitive truth that can achieve significant proof-theoretic strength has received much attention in the recent literature. Solomon Feferman [Fef84] famously claimed that 'nothing like sustained ordinary reasoning can be carried out' in the standard non-classical systems that support strong forms of inter-substitutivity of A and 'A is true' that are sufficient to generate paradoxical phenomena together with classical logic. One way of understanding this claim is by measuring how much mathematics can be encoded in such systems. Since the strength of mathematical systems (whether classical or non-classical) is traditionally measured in terms of the ordinals that can be well-ordered by them, the ordinal analysis of non-classical systems of truth becomes relevant.

We are mainly interested in the proof-theoretic analysis of non-classical systems inspired by fixed-point semantics [Kri75]. Since fixed-point semantics has nice axiomatizations, both classical and non-classical, it represents a particularly convenient arena to measure the impact of weakening the logic on proof-theoretic strength. The axiomatization of fixed-point semantics in classical logic – a.k.a. **KF** – is known to have the proof-theoretic ordinal $\varphi_{\varepsilon_0}0$ [Fef91, Can89],[1] Halbach and Horsten have proposed in [HH06] a non-classical axiomatization, known as **PKF**, and showed that it has proof-theoretic ordinal $\varphi_\omega 0$. There have been some attempts to overcome this mismatch in strength on the non-classical side. [Nic17] showed that even without expanding

[1]Or Γ_0, depending on whether one focuses on a version of the theory with or without suitable open-ended substitution rule schemata.

the logical resources of theory, **PKF** can be extended with suitable instances of transfinite induction to recover all classical true theorems of **KF**. [FHN17] show that a simple theory featuring non-classical initial sequents of the form $A \Rightarrow \mathtt{Tr}\ulcorner A \urcorner$ and $\mathtt{Tr}\ulcorner A \urcorner \Rightarrow A$ can be closed under special reflection principles to recover the arithmetical strength of **PKF** and **KF**. More recently, [Fie20] has shown that, by enlarging the primitive concepts of **PKF** with a predicate for 'classicality', one can achieve the proof-theoretic strength of **KF** in both the schematic and non-schematic versions.

In the paper we explore a different option, which in a sense completes the picture above. We enlarge the standard four-valued logic of **PKF** with a new conditional, which is based on the logic **HYPE** recently proposed by [Lei19]. The conditional has several features that resemble an intuitionistic conditional, but its weaker interaction with the **FDE** -negation makes it possible to sustain the intersubstitutivity of A and 'A is true' for sentences not containing the conditional. This extended theory, that we call **KFL**, is shown to be proof-theoretically equivalent with **KF**. Its extension with a schematic substitution rule, called **KFL***, is shown to be proof-theoretically equivalent to the schematic extension of **KF** – called $\mathtt{Ref}^*(\mathtt{PA}(P))$ in [Fef91].

In particular, we show that the conditional proper of the logic **HYPE** enables one to mimic, when carefully handled, the standard lower bound proofs by Gentzen and Feferman-Schütte for transfinite induction in classical arithmetic (Theorem 1) and predicative analysis (Proposition 4), respectively. This enables us to define, in our theories **KFL** and **KFL***, ramified truth predicates indexed by ordinals smaller than ε_0 (Corollary 5) and Γ_0 (Corollary 7). Moreover, the proof-theoretic analysis of **KFL** and **KFL*** is completed by showing that their truth predicates can be suitably interpreted in their classical counterparts **KF** and $\mathtt{Ref}^*(\mathtt{PA}(P))$ without altering the arithmetical vocabulary (Propositions 2 and 5).

2. HYPE

In this section we will present the logical basis of our systems of truth. We will work with a sequent calculus variant of the logic **HYPE** introduced by Leitgeb in [Lei19]. Essentially, the calculus is obtained by extending First-Degree Entailment with an intuitionistic conditional and rules for it in a multi-conclusion style.

2.1. $\mathbf{G1h_{cd}}$. We present a multiconclusion system based on a multi-conclusion calculus for intuitionistic logic:[2] we call it $\mathbf{G1h_{cd}}$ for Gentzen system for the logic **HYPE** with constant domains. Sequents are understood as multisets. We work with a language whose logical symbols are $\neg$, $\vee$, $\rightarrow$, $\forall$, $\bot$. For $\Gamma = \gamma_1, \ldots, \gamma_n$ a multiset, $\neg\Gamma$ is the multiset $\neg\gamma_1, \ldots, \neg\gamma_n$. $\wedge, \exists$ can be defined as usual and $\top$ is defined as $\neg\bot$. Moreover, we can define 'intuitionistic' negation $\sim A$ as $A \rightarrow \bot$, the material conditional $A \supset B$ as $\neg A \vee B$, and material equivalence $A \equiv B$ as $(A \supset B) \wedge (B \supset A)$.

[2]This system goes back to Maehara's version used in Takeuti [Tak87] p.52f and Dragalin's system used in Negri and Plato [NP01] p.108f.

The system $\mathbf{G1h_{cd}}$ consists of the following initial sequents and rules:

$$(\mathtt{ID}_p)\quad A \Rightarrow A \qquad\qquad (\mathtt{L}\bot)\quad \bot \Rightarrow .$$

$$(\mathtt{Cut})\ \frac{\Gamma \Rightarrow \Delta, A \qquad A, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}$$

$$(\mathtt{LW})\ \frac{\Gamma \Rightarrow \Delta}{A, \Gamma \Rightarrow \Delta} \qquad (\mathtt{RW})\ \frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, A}$$

$$(\mathtt{LC})\ \frac{A, A, \Gamma \Rightarrow \Delta}{A, \Gamma \Rightarrow \Delta} \qquad (\mathtt{RC})\ \frac{\Gamma \Rightarrow \Delta, A, A}{\Gamma \Rightarrow \Delta, A}$$

$$(\mathtt{L}\vee)\ \frac{A, \Gamma \Rightarrow \Delta \qquad B, \Gamma \Rightarrow \Delta}{A \vee B, \Gamma \Rightarrow \Delta} \qquad (\mathtt{R}\vee)\ \frac{\Gamma \Rightarrow A, B, \Delta}{\Gamma \Rightarrow A \vee B, \Delta}$$

$$(\mathtt{L}\to)\ \frac{\Gamma \Rightarrow \Delta, A \qquad B, \Gamma \Rightarrow \Delta}{A \to B, \Gamma \Rightarrow \Delta} \qquad (\mathtt{R}\to)\ \frac{\Gamma, A \Rightarrow B}{\Gamma \Rightarrow A \to B, \Delta}$$

$$(\mathtt{ConCp})\ \frac{\Gamma \Rightarrow \neg\Delta}{\Delta \Rightarrow \neg\Gamma} \qquad (\mathtt{ClCp})\ \frac{\neg\Gamma \Rightarrow \Delta}{\neg\Delta \Rightarrow \Gamma}$$

$$(\mathtt{L}\forall\ )\ \frac{A(t), \Gamma \Rightarrow \Delta}{\forall x A, \Gamma \Rightarrow \Delta} \qquad (\mathtt{R}\forall)\ \frac{\Gamma \Rightarrow \Delta, A(y)}{\Gamma \Rightarrow \Delta, \forall x A}\ y \notin FV(\Gamma, \Delta, \forall x A)$$

We write $\mathtt{rk}(A)$ for the logical complexity of A, defined as the number of nodes in the longest branch of its syntactic tree. For a derivation d we let

- $\mathtt{hgt}(d) := \mathtt{sup}_{i<n}\{\mathtt{hgt}(d_i) + 1 \mid\ d_i$ an immediate subderivation of $d\}$ (the height of the derivation), where $d_0, ..., d_n$ are the immediate subderivations of d (the *cut-rank* of d).

On occasion, we write $\vdash^n \Gamma \Rightarrow \Delta$ for 'there exists a derivation d of $\Gamma \Rightarrow \Delta$ with $\mathtt{hgt}(d) \leq n$'.

The next lemma collects some basic facts about $\mathbf{G1h_{cd}}$.

Lemma 1.

(i) *The sequents* $\Rightarrow \top$, $A \Rightarrow \neg\neg A$, $\neg\neg A \Rightarrow A$, *are derivable in* $\mathbf{G1h_{cd}}$.

(ii) *The rule of contraposition*

$$\frac{\Gamma \Rightarrow \Delta}{\neg\Delta \Rightarrow \neg\Gamma}$$

is admissible in $\mathbf{G1h_{cd}}$

(iii) *The following rules are admissible in* $\mathbf{G1h_{cd}}$*:*

$$(\mathtt{L}\wedge)\ \frac{A, B, \Gamma \Rightarrow \Delta}{A \wedge B, \Gamma \Rightarrow \Delta} \qquad (\mathtt{R}\wedge)\ \frac{\Gamma \Rightarrow A, \Delta \qquad \Gamma \Rightarrow B, \Delta}{\Gamma \Rightarrow A \wedge B, \Delta}$$

$$(\mathtt{L}\exists)\ \frac{A(y), \Gamma \Rightarrow \Delta}{\exists x A, \Gamma \Rightarrow \Delta}\ y \notin \mathtt{FV}(\Gamma, \Delta, \exists x A) \qquad (\mathtt{R}\exists)\ \frac{\Gamma \Rightarrow \Delta, A(t)}{\Gamma \Rightarrow \Delta, \exists x A}$$

(iv) *Intersubstitutivity: If* $\chi \Rightarrow \chi'$ *and* $\chi' \Rightarrow \chi$*, as well as* ψ *are derivable in* $\mathbf{G1h_{cd}}$*, then* $\psi(\chi'/\chi)$ *is derivable, where* $\psi(\chi'/\chi)$ *is obtained by replacing all occurrences of* χ *in* ψ *by* χ'*.*

We opted for this specific formulation of $\mathbf{G1h_{cd}}$ mainly for reasons of simplicity. From a proof-theoretic point of view, the calculus has some drawbacks even at the propositional level, as the rules `ConCp` and `ClCp` compromise the induction needed for cut-elimination. This problem could be solved, in the propositional case, by splitting the contraposition rule on a case by case manner [Fis20]. However, when one moves to the quantificational system, there are deeper problems. The same counterexample that is employed to show that cut is not admissible in systems of intuitionistic logic with constant domains can be employed for the systems we are investigating.[3] Therefore, since cut elimination is beyond reach anyway for constant domains quantificational rules, we opt for a more compact presentation of $\mathbf{G1h_{cd}}$ that fits nicely our purpose of extending it with arithmetic and truth rules.

2.2. **Equality.** $\mathbf{G1h_{cd}}$ can be extended with a theory of equality. $\mathbf{G1h^{=}_{cd}}$ is obtained by adding to $\mathbf{G1h_{cd}}$ the following initial sequents for equality.

(`Ref`) $\Rightarrow t = t$

(`Rep`) $s = t, A(s) \Rightarrow A(t)$

In $\mathbf{G1h^{=}_{cd}}$, given `ConCp`, we can recover the classical principles for identity statements.

Lemma 2. $\Rightarrow s = t, \neg s = t$ *and* $s = t, \neg s = t \Rightarrow$ *are derivable.*

Proof. We use the identity sequents:

$$\dfrac{\dfrac{\dfrac{s = t, \neg s = t \Rightarrow \neg t = t \qquad \dfrac{\Rightarrow t = t}{\neg t = t \Rightarrow}}{s = t, \neg s = t \Rightarrow}}{\Rightarrow \neg s = t, \neg\neg s = t}}{\Rightarrow \neg s = t, s = t}$$

□

Lemma 2 reveals some subtle issues concerning the treatment of identity in subclassical logics generally employed to deal with semantical paradoxes. It tells us that identity is essentially treated as a classical notion in $\mathbf{G1h^{=}_{cd}}$. To obtain a similar phenomenon in absence of `ConCp` and `ClCp`, one would have to add the counterpositives of `Rep` and `Ref` to the system. A non-classical treatment for identity would require some non-trivial changes to `Rep` and `Ref`. That identity is a classical notion is perfectly in line with our framework, in which identity is a non-semantic notion akin to mathematical notions.

[3]See for example López-Escobar [LE83].

2.3. **Semantics.** In this section we present the semantics of $\mathbf{G1h_{cd}}$ (and therefore of **HYPE**) and sketch its completeness with respect to it. We follow a simplification of the semantics in Leitgeb [Lei19] suggested by Speranski [Spe20]. Speranski connects the **HYPE**-models with Routley semantics. A Routley frame $\mathfrak{F}$ is a triple $\langle W, \leq, * \rangle$, where:

(1) W is a non-empty set (we can think of them as states);
(2) $\leq$ is a preorder;
(3) $*$ is a function from W to W, which is:
   (a) antimonotone, i.e. for all $w, v \in W$, if $w \leq v$, then $v^* \leq w^*$;
   (b) involutive, i.e. for all $w \in W$, $w^{**} = w$.

A constant domain model $\mathfrak{M}$ for **HYPE** is a triple $(\mathfrak{F}, D, I)$ where: $\mathfrak{F}$ is a Routley frame, D is a non-empty set (the domain of the model), and I is an interpretation function. In particular, I assigns to every constant c an element of D and it associates with each state w and n-place predicate P a set $P^w \subseteq D^n$. The constants are then interpreted rigidly and, although domains do not grow, we impose the following hereditariness condition: for all $v, w \in W$, if $v \leq w$, then for all predicates P, $P^v \subseteq P^w$.

Let $\mathfrak{M}$ be a constant domain model, $w \in W$ and $\sigma\colon \mathrm{VAR} \to D$ a variable assignment on D, then the forcing relation $\mathfrak{M}, w, \sigma \Vdash A$ is defined inductively:

$\mathfrak{M}, w, \sigma \Vdash P(x_1, ..., x_n)$ iff $(\sigma(x_1), ..., \sigma(x_n)) \in P^w$;

$\mathfrak{M}, w, \sigma \Vdash \neg A$ iff $\mathfrak{M}, w^*, \sigma \nVdash A$;

$\mathfrak{M}, w, \sigma \Vdash A \vee B$ iff $\mathfrak{M}, w, \sigma \Vdash A$ or $\mathfrak{M}, w, \sigma \Vdash B$;

$\mathfrak{M}, w, \sigma \Vdash A \to B$ iff for all v, with $w \leq v$, if $\mathfrak{M}, v, \sigma \Vdash A$, then $\mathfrak{M}, v, \sigma \Vdash B$;

$\mathfrak{M}, w, \sigma \Vdash \forall x A$ iff for all x-variants σ' of $\sigma, \mathfrak{M}, w, \sigma' \Vdash A$;

$\mathfrak{M}, w, \sigma \Vdash \top$ and $\mathfrak{M}, w, \sigma \nVdash \bot$.

Finally, we define logical consequence. We write, for Γ, Δ sets of sentences:

- $\mathfrak{M}, w \Vdash \Gamma \Rightarrow \Delta$ iff: if $\mathfrak{M}, w \Vdash \gamma$ for all $\gamma \in \Gamma$, then $\mathfrak{M}, w \Vdash \delta$ for some $\delta \in \Delta$;
- $\Gamma \Vdash \Delta$ iff for all $\mathcal{M}, w$: $\mathfrak{M}, w \Vdash \Gamma \Rightarrow \Delta$.

$\mathbf{G1h_{cd}}$ is equivalent to the following Hilbert-style system $\mathbf{QN}^\circ$ featuring the axiom schemata:

$$
\begin{array}{ll}
A \to (B \to C) & A \to (B \to C) \to ((A \to B) \to (B \to C)) \\
A \wedge B \to A & A \wedge B \to B \\
A \to A \vee B & B \to A \vee B \\
A \to (B \to A \wedge B) & (A \to C) \to ((B \to C) \to (A \vee B \to C)) \\
\neg\neg A \to A & A \to \neg\neg A \\
\forall x A \to A(t) & A(t) \to \exists x A
\end{array}
$$

and the following rules of inference:

$$\frac{A \qquad A \to B}{B}\,(\text{MP}) \qquad\qquad \frac{A \to B}{\neg B \to \neg A}\,(\text{CP})$$

$$\frac{A \to B(x)}{A \to \forall x B}\; x \text{ not free in } A \qquad\qquad \frac{A(x) \to B}{\exists x A \to B}\; x \text{ not free in } B$$

Our system $\mathbf{G1h_{cd}}$ is equivalent to $\mathbf{QN}^{\circ}$. We know in fact that all the axioms of $\mathbf{QN}^{\circ}$ except for the double negation axioms are intuitionistically valid. So with our Lemma 1 we can derive all the axioms of $\mathbf{QN}^{\circ}$ as well as the admissibility of contraposition. Rules for quantifiers are easily established in $\mathbf{G1h_{cd}}$. For the other direction a proof on the length of the derivations is sufficient and the fact that in $\mathbf{QN}^{\circ}$ the deduction theorem holds simplifies the proof. Therefore, we have:

Lemma 3. $\mathbf{G1h_{cd}} \vdash \Gamma \Rightarrow \Delta$ *iff* $\mathbf{QN}^{\circ} \vdash \bigwedge \Gamma \to \bigvee \Delta$.

Speranski [Spe20] establishes a strong completeness result (for countable signatures) for $\mathbf{QN}^{\circ}$. Speranski uses a Henkin-style proof similar to the strategy employed in Gabbay et al. [GSS09, §7.2] for intuitionistic logic with constant domains. Leitgeb [Lei19] establishes a (weak) completeness proof for his Hilbert style system based on the work of Görnemann [Gör71]. By Lemma 3 we can employ Speranski's completeness result for our system $\mathbf{G1h_{cd}}$ with respect to Routley semantics:

Proposition 1 (Completeness of $\mathbf{G1h_{cd}}$ [Spe20])**.** $\Gamma \Vdash \Delta$ *iff there is a finite* $\Delta_0 \subseteq \Delta$*, such that* $\Gamma \vdash_{\mathbf{QN}^{\circ}} \Delta_0$.

2.4. **HYPE and recapture.** One of the desirable properties of the non-classical logics employed in the debate on semantic paradoxes is the capability of recapturing classical reasoning in domains where there is no risk of paradoxicality, such as mathematics – see e.g. [Fie08].[4]

The following lemma summarizes the recapture properties of $\mathbf{G1h_{cd}}$ and extensions thereof. It essentially states that, in systems based on $\mathbf{G1h_{cd}}$, once we restrict our attention to a fragment of the language satisfying the excluded middle and/or explosion, the native **HYPE**-negation and conditional, as well as the defined intuitionistic negation, all behave fully classically.

Lemma 4.

[4]This form of recapture is a slightly different phenomenon than a direct, provability preserving, translation of the entire language of one theory in the other, as it happens for instance in the famous Gödel-Gentzen translation or the S4 interpretations of classical in intuitionistic logic, or intutionistic logic in modal logic respectively. While those translations provide a method to reinterpret the logical vocabulary – by keeping the non-logical vocabulary fixed – in a provability-preserving way, recapture strategies typically show that, for a specific fragment of its language, the non-classical theory behaves according to the rules of classical logic. For instance, that a non-classical theory of truth behaves fully classically if one restricts her attention to the truth-free language. To carry on with the analogy with the relationships between classical and intuitionistic logic, recapture strategies are much closer to the identity between the Δ_1-fragments of classical and intuitionistic arithmetic.

(i) *The following rules are admissible in extensions of* $\mathbf{G1h_{cd}}$*:*

$$\dfrac{\Rightarrow A, \neg A \qquad \Gamma, A \Rightarrow \Delta}{\Gamma \Rightarrow \neg A, \Delta} \qquad\qquad \dfrac{A, \neg A \Rightarrow \qquad \Gamma \Rightarrow A, \Delta}{\Gamma, \neg A \Rightarrow \Delta}$$

$$\dfrac{\Rightarrow A, \neg A \qquad \Gamma, A \Rightarrow B, \Delta}{\Gamma \Rightarrow A \to B, \Delta}$$

$$\dfrac{A, \neg A \Rightarrow}{\neg A \Rightarrow A \to \bot} \qquad\qquad \dfrac{A, \neg A \Rightarrow}{A \to \bot \Rightarrow \neg A}$$

$$\dfrac{\Rightarrow A, \neg A}{A \to B \Rightarrow A \supset B} \qquad\qquad \dfrac{\Rightarrow A, \neg A}{A \supset B \Rightarrow A \to B}$$

(ii) *The previous fact can be used to show, by an induction on* $\mathtt{rk}(A)$*, that* $\Rightarrow A, \neg A$ *is derivable for any formula whenever* $\Rightarrow P, \neg P$ *is derivable for any atomic* P *in* A*.*

Proof. We prove the claims for the crucial cases in which a conditional is involved:

For (i):

$$\dfrac{\dfrac{\dfrac{\dfrac{\Rightarrow A, \neg A}{\Gamma \Rightarrow A, \neg A, B, \Delta} \qquad \dfrac{\Gamma, A \Rightarrow B, \Delta}{\Gamma, A \Rightarrow \neg A, B, \Delta}}{\Gamma \Rightarrow \neg A, B, \Delta}}{\Gamma \Rightarrow B, \neg A, A \to B, \Delta} \qquad \dfrac{\dfrac{B, A \Rightarrow B}{B \Rightarrow A \to B}}{\Gamma, B \Rightarrow \neg A, A \to B, \Delta}}{\Gamma \Rightarrow \neg A, A \to B, \Delta} \qquad \dfrac{\dfrac{\dfrac{A, \neg A \Rightarrow}{A, \neg A \Rightarrow B}}{\neg A \Rightarrow A \to B}}{\Gamma, \neg A \Rightarrow A \to B, \Delta}}{\Gamma \Rightarrow A \to B, \Delta}$$

For (ii):

$$\dfrac{\dfrac{\dfrac{\neg A, A \Rightarrow B}{\neg A \Rightarrow A \to B}}{\neg(A \to B) \Rightarrow A} \qquad \dfrac{\dfrac{\dfrac{B, A \Rightarrow B}{B \Rightarrow A \to B}}{\neg(A \to B) \Rightarrow \neg B}}{B, \neg(A \to B) \Rightarrow}}{A \to B, \neg(A \to B) \Rightarrow}$$

□

Remark 1. The induction involved in Lemma 4(ii) does not go through in intuitionistic logic with the **HYPE**-negation ¬ replaced by the intuitionistic negation.

Corollary 1. *Let* $\mathcal{L}$ *be a language based on classical predicate symbols, i.e. for all* P *in* $\mathcal{L}$*,* $\Rightarrow P(t), \neg P(t)$ *and* $P(t), \neg P(t) \Rightarrow$ *for all terms* t*, then for all formulas* A *of* $\mathcal{L}$*,* $\Rightarrow A, \neg A$ *and* $A, \neg A \Rightarrow$ *are derivable.*

3. Arithmetic in HYPE

We work with a suitable expansion of the usual signature $\{0, \mathtt{S}, +, \times\}$ by finitely many function symbols for selected primitive recursive functions suitable for a smooth representation of syntax theory. We call this language $\mathcal{L}_{\mathbb{N}}^{\to}$. We will also make use of the →-free fragment of the language of arithmetic, which we label as $\mathcal{L}_{\mathbb{N}}$. Our base theory will then be obtained by adding, to the basic axioms for $0, \mathtt{S}, +, \times$ (axioms **Q**1-2, **Q**4-7 of [HP93]), the recursive clauses for these additional function symbols. The resulting system will be called $\mathbf{HYA}^-$.

In the following, the role of rule and axiom schemata will be crucial. It will be particularly important to keep track of the classes of instances of a particular schema, and therefore we will always relativize schemata to specific languages and understand the schema as the set of all its instances in that language. For example, in the case of the induction axioms we use the label $\mathtt{IND}^{\rightarrow}(\mathcal{L})$ to refer to the set of all sequents of the form

$$(\mathtt{IND}^{\rightarrow}(\mathcal{L})) \qquad \Rightarrow A(0) \wedge \forall x(A(x) \rightarrow A(x+1)) \rightarrow \forall x A(x),$$

where A is a formula of $\mathcal{L}$. Similarly, induction rules $\mathtt{IND}^{\mathtt{R}}(\mathcal{L})$ will refer to all rule instances

$$\frac{\Gamma, A(x) \Rightarrow A(x+1), \Delta}{\Gamma, A(0) \Rightarrow A(t), \Delta}\ (\mathtt{IND}^{\mathtt{R}}(\mathcal{L}))$$

for A a formula of $\mathcal{L}$.

We call the extension of $\mathbf{HYA}^-$ by the (unrestricted) induction rule **HYA**. We can easily see that **HYA** is equivalent to Peano Arithmetic **PA**. This relies on the recapture properties of our logic. Especially interesting is that, in **HYA**, we have a formulation of induction as a sequent, which is equivalent to the rule formulation for formulas A containing only classical vocabulary. By the properties stated in Lemma 4, we get:

Lemma 5. *Let $\mathcal{L} \supseteq \mathcal{L}_{\mathbb{N}}^{\rightarrow}$. Over $\mathbf{HYA}^-$: $\mathtt{IND}^{\mathtt{R}}(\mathcal{L})$ and $\mathtt{IND}^{\rightarrow}(\mathcal{L})$ are equivalent when restricted to formulas A such that $\Rightarrow A, \neg A$.*

Since for $A \in \mathcal{L}_{\mathbb{N}}^{\rightarrow}$, $\Rightarrow A, \neg A$ and $A, \neg A \Rightarrow$ are derivable in $\mathbf{G1h_{cd}}$, we have the immediate corollary that:

Corollary 2. **HYA** *is equivalent to* **PA**.

3.1. **Ordinals and transfinite induction.** Our notational conventions for schemata generalize to schemata other than induction. A prominent role in the paper will be played by *transfinite induction schemata*. In order to introduce them, we need to assume a notation system $(\mathtt{OT}, \prec)$ for ordinals up to the Feferman-Schütte ordinal Γ_0 as it can be found, for instance, in [Poh09, Ch. 2]. $\mathtt{OT}$ is a primitive recursive set of ordinal codes and $\prec$ a primitive recursive relation on $\mathtt{OT}$ that is isomorphic to the usual ordering of ordinals up to Γ_0. We distinguish between fixed ordinal codes, which we denote with $\alpha, \beta, \gamma \ldots$, and $\zeta, \eta, \theta, \xi, \ldots$ as abbreviations for variables ranging over elements of $\mathtt{OT}$. Our representation of ordinals satisfies all standard properties. In particular, we will make implicit use of such properties that one can find in [TS00], p.322.

We will make extensive use of the following abbreviations. We call a formula *progressive* if it is preserved upwards by the ordinals:

$$\mathtt{Prog}(A) := \forall \eta(\forall \zeta \prec \eta\, A(\zeta) \rightarrow A(\eta))$$

This formulation is **HYA**-equivalent to a formulation as a sequent $\forall\zeta \prec \eta\, A(\zeta) \Rightarrow A(\eta)$. Moreover, if $A(x) \vee \neg A(x)$ is provable, then $\mathtt{Prog}(A)$ is **HYA**-equivalent to:

(1) $$\forall\eta(\forall\zeta \prec \eta\, A(\zeta) \supset A(\eta)).$$

Transfinite induction up to the ordinal α ($\prec \Gamma_0$) will be formulated as the following rule:

$$\mathtt{TI}_\alpha(A) := \frac{\Gamma, \forall\zeta \prec \eta\, A(\zeta) \Rightarrow A(\eta), \Delta}{\Gamma \Rightarrow \forall\xi \prec \alpha\, A(\xi), \Delta}$$

$\mathtt{TI}_\alpha(\mathcal{L})$ is short for $\mathtt{TI}_\alpha(A)$ for every formula A of the language $\mathcal{L}$. $\mathtt{TI}_{<\alpha}(\mathcal{L})$ is short for $\mathtt{TI}_\beta(\mathcal{L})$ for all $\beta \prec \alpha$.

We define recursively the function ω_n: $\omega_0 = 1$, $\omega_{n+1} = \omega^{\omega_n}$.

3.2. **Transfinite induction and non-classical predicates.** Our main purpose in this paper is to study the proof-theoretic properties of extensions of **HYA** with additional predicates that may not behave classically – i.e. they may not satisfy Lemma 5. In fact, in the case of the pure arithmetical language, Lemma 5 gives us immediately that **HYA** derives $\mathtt{TI}_{<\varepsilon_0}(\mathcal{L}_{\mathbb{N}}^{\rightarrow})$. In this section we show directly that Gentzen's original proof of $\mathtt{TI}_{<\varepsilon_0}(\mathcal{L}_{\mathbb{N}}^{\rightarrow})$ can be carried out in **HYA** for suitable extensions of $\mathcal{L}_{\mathbb{N}}^{\rightarrow}$.[5]

Theorem 1. *Let $\mathcal{L}^+$ be a language expansion of $\mathcal{L}_{\mathbb{N}}^{\rightarrow}$ by finitely many predicate symbols. Then* HYA $\vdash \mathtt{TI}_{<\varepsilon_0}(\mathcal{L}^+)$.

The rest of this subsection will be devoted to the proof of Theorem 1, which will involve several preliminary lemmata.

A key ingredient of Gentzen's proof – which will also play an important role in subsequent sections – is the Gentzen's jump formula:

$$A^+(\theta) := \forall\xi(\forall\eta(\eta \prec \xi \to A(\eta)) \to \forall\eta(\eta \prec \xi + \omega^\theta \to A(\eta)))$$

Lemma 6. *For any $A \in \mathcal{L}^+$, if $\mathtt{Prog}(A)$ is derivable in* **HYA**, *then $\mathtt{Prog}(A^+)$ is derivable in* **HYA**.

Proof. We want to show $\mathtt{Prog}(A^+)$, i.e. $\forall\eta(\forall\zeta \prec \eta\, A^+(\zeta) \to A^+(\eta))$. Informally we make a case distinction: Either $\theta = 0$ or $\theta \succ 0$.

Case 1: If $\theta = 0$, then

(2) $$\theta = 0, \eta \prec \xi + \omega^\theta \Rightarrow \eta \prec \xi \vee \eta = \xi.$$

Since $\mathtt{Prog}(A)$, we have

(3) $$\forall\delta\,(\delta \prec \xi \to A(\delta)), \eta \prec \xi \Rightarrow A(\eta)$$

(4) $$\forall\delta\,(\delta \prec \xi \to A(\delta)), \eta = \xi \Rightarrow A(\eta)$$

[5]Troelstra & Schwichtenberg [TS00] established that the Gentzen proof can be carried out in the minimal $\to\forall\bot$ fragment of $\mathtt{IL}$.

By (2) and $\texttt{Cut}$, we obtain

$$\theta = 0, \eta \prec \xi + \omega^{\theta} \Rightarrow A(\eta).$$

Therefore, an application of $(\texttt{R}\rightarrow)$ and weakening yield

$$\theta = 0 \Rightarrow \forall \zeta \prec \eta\, (A^{+}(\zeta)) \rightarrow A^{+}(\eta)$$

Case 2: $\theta \succ 0$. Then by a derivable version of Cantor's Normal Form Theorem:

$$(\dagger) \qquad \theta \succ 0, \eta \prec \xi + \omega^{\theta} \Rightarrow \exists n\, \exists \theta_0 \prec \theta(\eta \prec \xi + \omega^{\theta_0} \cdot n).$$

By induction on n we will show that under the assumption $\forall \zeta \prec \theta\, A^{+}(\zeta)$

$$\theta_0 \prec \theta \Rightarrow \forall \eta(\eta \prec \xi + \omega^{\theta_0} \cdot n \rightarrow A(\eta)).$$

The base case is straightforward because the following is trivially derivable (by property ($\texttt{ord6}$)):

$$(5) \qquad \forall \eta \prec \xi\, A(\eta) \Rightarrow (\forall \eta \prec \xi + \omega^{\theta_0} \cdot 0)\, A(\eta).$$

For the induction step, we start by noticing that we can derive the following,

$$(6) \qquad A^{+}(\theta_0), \forall \eta \prec \xi + \omega^{\theta_0} \cdot n\, A(\eta) \Rightarrow \forall \eta \prec \xi + \omega^{\theta_0} \cdot (n+1)\, A(\eta)$$

which entails, since induction for ordinal notations up to ω is provable in **PA**,

$$(7) \qquad A^{+}(\theta_0) \Rightarrow \forall n \forall \eta \prec \xi + \omega^{\theta_0} \cdot n\, A(\eta).$$

By cut and the definition of $A^{+}(\theta_0)$, from (7) we obtain:

$$(8) \qquad \forall \eta \prec \xi\, A(\eta), \forall \zeta \prec \theta\, A^{+}(\zeta), \theta_0 \prec \theta \Rightarrow \forall n \forall \eta \prec \xi + \omega^{\theta_0} \cdot n\, A(\eta)$$

Therefore, the logical rules of $\texttt{HYPE}$ give us:

$$(9) \qquad \theta \succ 0, \forall \eta \prec \xi\, A(\eta), \forall \zeta \prec \theta\, A^{+}(\zeta), \exists n\, \exists \theta_0 \prec \alpha(\eta \prec \xi + \omega^{\theta_0} \cdot n) \Rightarrow A(\eta),$$

which in turn by (†) gives us:

$$(10) \qquad \theta \succ 0, \forall \eta \prec \xi\, A(\eta), \forall \zeta \prec \theta\, A^{+}(\zeta) \Rightarrow \forall \eta \prec \xi + \omega^{\theta}\, A(\eta)$$

By applying the rule $(\texttt{R}\rightarrow)$ we finally get $\texttt{Prog}(A^{+})$.

□

The progressiveness of Gentzen's jump formula enables us then to establish:

Lemma 7. *If* $\texttt{TI}_{\alpha}(\mathcal{L}^{+})$ *is admissible in* **HYA**, *then* $\texttt{TI}_{\omega^{\alpha}}(\mathcal{L}^{+})$ *is admissible in* **HYA**.[6]

Proof. What we want to show is, in fact, that if the following rule of transfinite induction is admissible in **HYA**

$$\texttt{TI}_{\alpha}(A) := \frac{\Gamma, \forall \zeta \prec \eta\, A(\zeta) \Rightarrow A(\eta), \Delta}{\Gamma \Rightarrow \forall \xi \prec \alpha\, A(\xi), \Delta} \text{ for any } A \in \mathcal{L}^{+}$$

[6] The notion of admissible rule that we employ is the one from [TS00, p. 76].

then also the following rule is admissible in **HYA**:

$$\mathtt{TI}_{\omega^\alpha}(A) := \frac{\Gamma, \forall\zeta \prec \eta\, A(\zeta) \Rightarrow A(\eta), \Delta}{\Gamma \Rightarrow \forall\xi \prec \omega^\alpha\, A(\xi), \Delta} \text{ for any } A \in \mathcal{L}^+$$

Thus, we assume that $\mathtt{TI}_\alpha(A)$ is admissible and $\mathtt{Prog}(A)$ is derivable for some arbitrary $A \in \mathcal{L}^+$. Then by our previous lemma $\mathtt{Prog}(A^+)$ is also derivable. By assumption we have $\mathtt{TI}_\alpha(A)$ for all $A \in \mathcal{L}^+$, especially for A^+:

$$\Rightarrow \forall\beta \prec \alpha(A^+(\beta)) \tag{11}$$

Therefore, by $\mathtt{Prog}$ (A^+) and $\mathtt{Cut}$:

$$\Rightarrow \forall\xi(\forall\eta \prec \xi\, A(\eta) \to \forall\eta \prec \xi + \omega^\alpha\, A(\eta)) \tag{12}$$

But also

$$\Rightarrow \forall\eta \prec 0\, A(\eta), \tag{13}$$

and therefore by (12), we obtain

$$\Rightarrow \forall\eta \prec \omega^\alpha A(\eta)$$

as desired. □

Corollary 3. *If A is such that* **HYA** *proves $A(x) \vee \neg A(x)$, we have that, if* **HYA** *proves the classical transfinite induction axiom schema for α*

$$(\forall\zeta \prec \eta A(\zeta) \supset A(\eta)) \supset \forall\xi \prec \alpha\, A(\xi),$$

then **HYA** *proves:*

$$(\forall\zeta \prec \eta A(\zeta) \supset A(\eta)) \supset \forall\xi \prec \omega^\alpha\, A(\xi).$$

All is set up to finally prove the main result of this section, the admissibility in **HYA** of the required schema of transfinite induction up to any ordinal $\alpha \prec \varepsilon_0$.

Proof of Theorem 1. The result follows immediately from the previous lemma. Since $\mathtt{TI}_{\omega_0}(A)$ is trivially derivable in **HYA**, the lemma tells us that $\mathtt{TI}_{\omega_n}(A)$, for each n, can be reached in finitely many proof steps.

□

4. Theory of Truth

In this section we introduce a theory of truth **KFL**. The theory is formulated in the language $\mathcal{L}_{\mathtt{Tr}}^{\to} := \mathcal{L}_{\mathbb{N}}^{\to} \cup \{\mathtt{Tr}\}$, where $\mathtt{Tr}$ is a unary predicate for truth. **KFL** will be a theory of truth for a $\to$-free language $\mathcal{L}_{\mathtt{Tr}}$, which is simply the $\to$-free fragment of $\mathcal{L}_{\mathtt{Tr}}^{\to}$. In **KFL**, the conditional $\to$ should be thought of as a theoretical device to formulate our semantic theory, and not as an object of the semantic investigation itself. Semantically (cf. §4.1), one should think of the conditional as a device to navigate between fixed point models in the sense of [Kri75].

Definition 1 (The language $\mathcal{L}_{\mathtt{Tr}}$)**.** *The logical symbols of $\mathcal{L}_{\mathtt{Tr}}$ are $\bot, \neg, \vee, \forall$. In addition, we have the identity symbol. Its non-logical vocabulary amounts to the arithmetical vocabulary of $\mathcal{L}_{\mathbb{N}}$ and the truth predicate* $\mathtt{Tr}$*.*

Definition 2 (The theory **KFL**)**.** **KFL** *extends* **HYA** *formulated in $\mathcal{L}_{\mathtt{Tr}}^{\rightarrow}$ – i.e. with induction rules extended to $\mathcal{L}_{\mathtt{Tr}}^{\rightarrow}$ – with the following truth initial sequents:*

$$(\mathbf{KFL}1) \qquad \mathtt{Cterm}_{\mathcal{L}_{\mathtt{Tr}}}(x) \wedge \mathtt{Cterm}_{\mathcal{L}_{\mathtt{Tr}}}(y) \Rightarrow \mathtt{Tr}(x \mathbin{\underset{.}{=}} y) \leftrightarrow \mathtt{val}(x) = \mathtt{val}(y)$$

$$(\mathbf{KFL}2) \qquad \Rightarrow \mathtt{Tr}(\ulcorner \mathtt{Tr}\dot{x} \urcorner) \leftrightarrow \mathtt{Tr}\, x$$

$$(\mathbf{KFL}3) \qquad \mathtt{Sent}_{\mathcal{L}_{\mathtt{Tr}}}(x) \Rightarrow \mathtt{Tr}\, \underset{.}{\neg}\, x \leftrightarrow \neg \mathtt{Tr}\, x$$

$$(\mathbf{KFL}4) \qquad \mathtt{Sent}_{\mathcal{L}_{\mathtt{Tr}}}(x) \wedge \mathtt{Sent}_{\mathcal{L}_{\mathtt{Tr}}}(y) \Rightarrow (\mathtt{Tr}(x \mathbin{\underset{.}{\wedge}} y) \leftrightarrow \mathtt{Tr}(x) \wedge \mathtt{Tr}(y))$$

$$(\mathbf{KFL}5) \qquad \mathtt{Sent}_{\mathcal{L}_{\mathtt{Tr}}}(x) \wedge \mathtt{Sent}_{\mathcal{L}_{\mathtt{Tr}}}(y) \Rightarrow (\mathtt{Tr}(x \mathbin{\underset{.}{\vee}} y) \leftrightarrow \mathtt{Tr}(x) \vee \mathtt{Tr}(y))$$

$$(\mathbf{KFL}6) \qquad \mathtt{Sent}_{\mathcal{L}_{\mathtt{Tr}}}(\underset{.}{\forall} v\, x) \wedge \mathtt{var}(v) \Rightarrow \mathtt{Tr}(\underset{.}{\forall} v\, x) \leftrightarrow \forall y(\mathtt{CTerm}_{\mathcal{L}_{\mathtt{Tr}}}(y) \rightarrow \mathtt{Tr}\, x(y/v))$$

$$(\mathbf{KFL}7) \qquad \mathtt{Sent}_{\mathcal{L}_{\mathtt{Tr}}}(\underset{.}{\forall} v\, x) \wedge \mathtt{var}(v) \Rightarrow \mathtt{Tr}(\underset{.}{\exists} v\, x) \leftrightarrow \exists y(\mathtt{CTerm}_{\mathcal{L}_{\mathtt{Tr}}}(y) \wedge \mathtt{Tr}\, x(y/v))$$

$$(\mathbf{KFL}8) \qquad \mathtt{Tr}\, x \Rightarrow \mathtt{Sent}_{\mathcal{L}_{\mathtt{Tr}}}(x)$$

According to Lemma 4 we have that $\bot$, $\supset$ and $\rightarrow$ obey the classical intro and elimination rules when the antecedent is a formula of $\mathcal{L}_{\mathbb{N}}^{\rightarrow}$.

Lemma 8. *The following are provable in* **KFL***:*

(i) $\mathtt{Sent}_{\mathcal{L}_{\mathtt{Tr}}}(x) \Rightarrow \mathtt{Tr}\ulcorner \neg \mathtt{Tr}\, \dot{x} \urcorner \leftrightarrow \mathtt{Tr}\, \underset{.}{\neg} x$

(ii) *For $A \in \mathcal{L}_{\mathtt{Tr}}$,* **KFL** $\vdash \mathtt{Tr}\ulcorner A \urcorner \leftrightarrow A$*.*

4.1. **Semantics.** The intended interpretation of our theory of truth is based on Kripke's fixed point semantics [Kri75].[7] The states of our model are going to be fixed-points of the usual monotone operator associated with the four-valued evaluation schema as stated in Visser [Vis84] and Woodruff [Woo84].

Let $\Phi \colon \mathcal{P}\omega \rightarrow \mathcal{P}\omega$ be the operator defined in Halbach [Hal14] Lemma 15.6. We let

$$\mathbb{W} := \{X \subseteq \omega \mid \Phi(X) = X\} \tag{14}$$

$$S \leq_{\mathbb{W}} S' :\Leftrightarrow S \subseteq S' \tag{15}$$

$$S^* = \omega \setminus \overline{S}, \text{ with } \overline{X} = \{\neg\varphi \mid \varphi \in X\} \tag{16}$$

$$\mathtt{Tr}^S := S \tag{17}$$

The intended full model $\mathfrak{M}_\Phi$ is then the **HYPE** model based on the frame $(\mathbb{W}, \leq_{\mathbb{W}}, *)$ with the constant domain ω and the interpretation of the truth predicate at each state given by the

[7] The intended model presented here is based on the model presented in Leitgeb [Lei19, §7].

extension of the corresponding fixed-point. The intended minimal model $\mathfrak{M}_{\Phi}^{\mathtt{min}}$ is then given by restricting the set of states to the minimal and maximal fixed points.

Lemma 9. *If* $\mathbf{KFL} \vdash \Gamma \Rightarrow \Delta$*, then* $\mathfrak{M}_{\Phi} \Vdash \Gamma \Rightarrow \Delta$.

4.2. **Proof Theory: lower bound.** We show that **KFL** can define (and therefore prove the well-foundedness of) Tarskian truth predicates for any $\alpha \prec \varepsilon_0$. By Feferman and Cantini's analyses of the proof theory of **KF** [Can89,Fef91], this entails that **KFL** can prove $\mathtt{TI}_{<\varphi_{\varepsilon_0}0}(\mathcal{L}_{\mathbb{N}})$.

We first define the Tarskian languages.

Definition 3. *For* $0 \leq \alpha < \Gamma_0$*, we let:*

$$\mathrm{Sent}_{\mathcal{L}_{\mathtt{Tr}}}(\overline{0}, x) :\leftrightarrow \mathrm{Sent}_{\mathcal{L}_{\mathbb{N}}}(x)$$

$$\begin{aligned}\mathrm{Sent}_{\mathcal{L}_{\mathtt{Tr}}}(\zeta+1, x) :\leftrightarrow\ & x = (s \dot{=} t) \vee \\ & (\exists y \leq x)(x = \dot{\mathtt{Tr}} y \wedge \mathrm{Sent}_{\mathcal{L}_{\mathtt{Tr}}}(\zeta, y)) \vee \\ & (\exists y \leq x)(x = (\dot{\neg} y) \wedge \mathrm{Sent}_{\mathcal{L}_{\mathtt{Tr}}}(\zeta+1, y)) \vee \\ & (\exists y \exists z \leq x)(x = (y \dot{\wedge} z) \wedge \mathrm{Sent}_{\mathcal{L}_{\mathtt{Tr}}}(\zeta+1, y) \wedge \mathrm{Sent}_{\mathcal{L}_{\mathtt{Tr}}}(\zeta+1, z)) \vee \\ & (\exists y \exists z \leq x)(x = (y \dot{\vee} z) \wedge \mathrm{Sent}_{\mathcal{L}_{\mathtt{Tr}}}(\zeta+1, y) \wedge \mathrm{Sent}_{\mathcal{L}_{\mathtt{Tr}}}(\zeta+1, z)) \vee \\ & (\exists v \exists y)(x = (\dot{\forall} v y) \wedge \mathrm{Sent}_{\mathcal{L}_{\mathtt{Tr}}}(\zeta+1, y))\end{aligned}$$

$$\mathrm{Sent}_{\mathcal{L}_{\mathtt{Tr}}}(\lambda, x) :\leftrightarrow \exists\, \zeta < \lambda\, \mathrm{Sent}_{\mathcal{L}_{\mathtt{Tr}}}(\zeta, x)$$

We then write:

$$\begin{aligned}\mathrm{Sent}^{<\alpha}_{\mathcal{L}_{\mathtt{Tr}}}(x) &:\leftrightarrow \exists \zeta \prec \alpha\, \mathrm{Sent}_{\mathcal{L}_{\mathtt{Tr}}}(\zeta, x) \\ \mathtt{Tr}_{\alpha}(x) &:\leftrightarrow \mathrm{Sent}^{<\alpha}_{\mathcal{L}_{\mathtt{Tr}}}(x) \wedge \mathtt{Tr}(x).\end{aligned}$$

As we mentioned, arithmetical vocabulary behaves classically in **KFL**.

Lemma 10. $\mathbf{KFL} \vdash \forall x(\mathtt{Sent}_{\mathcal{L}_{\mathbb{N}}}(x) \rightarrow \mathtt{Tr}x \vee \mathtt{Tr}\dot{\neg}x)$.

Proof. By formal induction on the complexity of the 'sentence' $x \in \mathcal{L}_{\mathbb{N}}$. □

The next two claims establish that the previous fact can then be extended to all Tarskian languages whose indices can be proved to be well-founded.

Lemma 11. $\mathbf{KFL} \vdash (\forall \zeta \prec \eta)(\mathtt{Sent}_{\mathcal{L}_{\mathtt{Tr}}}(\zeta, x) \rightarrow \mathtt{Tr}x \vee \mathtt{Tr}\dot{\neg}x) \Rightarrow \mathtt{Sent}_{\mathcal{L}_{\mathtt{Tr}}}(\eta, x) \rightarrow \mathtt{Tr}x \vee \mathtt{Tr}\dot{\neg}x$.

Proof. Provably in **KFL**, $\eta \in \mathtt{OT}$ is either 0, or a successor ordinal, or a limit. By arguing informally in **KFL**, we show that the statement of the lemma holds, thereby establishing the claim. Lemma 10 gives us the base case. □

By Theorem 1, we obtain:

Corollary 4. *For any* $\alpha \prec \varepsilon_0$, $\mathbf{KFL} \vdash \forall x\, (\mathtt{Sent}_{\mathcal{L}_{\mathtt{Tr}}}(\alpha, x) \rightarrow \mathtt{Tr}x \vee \mathtt{Tr}\dot{\neg}x)$.

Since, by Theorem 1, **KFL** proves transfinite induction up to ordinals smaller than ε_0, it follows that we are able to climb up the Tarskian hierarchy of languages for iterations of the truth predicate up to ε_0. For $\alpha < \Gamma_0$, $\mathbf{RT}_{<\alpha}$ refers to the theory of ramified truth predicates up to α, as defined in [Hal14, §9.1].

Corollary 5. **KFL** *defines the truth predicates of* $\mathbf{RT}_{<\alpha}$*, for* $\alpha \prec \varepsilon_0$*. Therefore,* **KFL** *proves* $\mathtt{TI}_{<\varphi_{\varepsilon_0}0}(\mathcal{L}_{\mathbb{N}})$*.*

4.3. **Proof Theory: upper bound.** We interpret **KFL** in the Kripke-Feferman system **KF**. For definiteness, we consider the version of **KF** formulated in a language $\mathcal{L}_{\mathbb{T},\mathbb{F}}$ featuring truth ($\mathbb{T}$) and falsity ($\mathbb{F}$) predicates. Such a version of **KF** is basically the one presented in [Can89, §2], but without the consistency axiom.

In order to interpret **KFL** into **KF**, we consider a two-layered translation that differentiates between the external and internal structures of $\mathcal{L}_{\mathtt{Tr}}^{\rightarrow}$-formulas. Essentially, the external translation fully commutes with negation, and translates the **HYPE** conditional as classical material implication, whereas in the internal one we treat negated truth ascriptions as falsity ascriptions and follow the positive inductive characterization. The internal translation therefore translates truth and non-truth of **KFL** as truth and falsity of **KF**, respectively.

Definition 4. *We define the translations* $\tau\colon \mathcal{L}_{\mathtt{Tr}} \to \mathcal{L}_{\mathbb{T},\mathbb{F}}$*, and* $\sigma\colon \mathcal{L}_{\mathtt{Tr}}^{\rightarrow} \to \mathcal{L}_{\mathbb{T},\mathbb{F}}$ *as follows:*

(i)

$$
\begin{aligned}
&(s = t)^\tau = s = t && (s \neq t)^\tau = s \neq t\\
&(\mathtt{Tr}t)^\tau = \mathbb{T}\tau(t) && (\neg\mathtt{Tr}t)^\tau = \mathbb{F}\tau(t)\\
&(\neg\neg\varphi)^\tau = (\varphi)^\tau && \\
&(\varphi \wedge \psi)^\tau = (\varphi)^\tau \wedge (\psi)^\tau && (\neg(\varphi \wedge \psi))^\tau = (\neg\varphi)^\tau \vee (\neg\psi)^\tau\\
&(\forall v\varphi)^\tau = \forall x(\varphi)^\tau(x/v) && (\neg(\forall v\varphi))^\tau = \exists x(\neg\varphi)^\tau(x/v).
\end{aligned}
$$

(ii)

$$
\begin{aligned}
&(s = t)^\sigma = s = t && \\
&(\mathtt{Tr}t)^\sigma = \mathbb{T}(t)^\tau && (\neg\mathtt{Tr}t)^\sigma = \mathbb{F}(t)^\tau\\
&(\neg\varphi)^\sigma = \neg\varphi^\sigma && \\
&(\varphi \wedge \psi)^\sigma = (\varphi)^\sigma \wedge (\psi)^\sigma && (\forall x\varphi)^\sigma = \forall x\varphi^\sigma\\
&(\varphi \to \psi)^\sigma = \neg(\varphi)^\sigma \vee (\psi)^\sigma &&
\end{aligned}
$$

KFL-proofs can then be turned, by the translation σ, into **KF**-proofs, as the next proposition shows.

Proposition 2. *If* $\mathbf{KFL} \vdash \Gamma \Rightarrow \Delta$*, then* $\mathbf{KF} \vdash (\bigwedge \Gamma \to \bigvee \Delta)^\sigma$*.*

Proof. The proof is by induction on the length of the proof in **KFL**. We consider a few key cases.

(**KFL**3): we require that (with $\equiv$ expressing material equivalence):

$$\mathbf{KF} \vdash \mathtt{Sent}_{\mathcal{L}_{\mathtt{Tr}}} \Rightarrow \mathbb{T}\tau(\underset{\cdot}{\neg} x) \equiv \mathbb{F}\tau(x). \tag{18}$$

However, this can be proved by formal induction on the complexity of x. □

The combination of Proposition 2 and Corollary 5 yields that **KF** and **KFL** have the same arithmetical theorems, and in particular they have the same proof-theoretic ordinal – cf. [Poh09, §6.7].

Corollary 6. $|\mathbf{KFL}| = |\mathbf{KF}| = \varphi_{\varepsilon_0}0$.

5. Schematic extension

5.1. **KFL*: rules and semantics.** In this section we study the schematic extension of **KFL** in the sense of [Fef91]. This is obtained by extending **KFL** with a special substitution rule that enables us to uniformly replace the special predicate P in arithmetical theorems $A(P)$ of our extended theory for arbitrary formulas of $\mathcal{L}^{\rightarrow}_{\mathtt{Tr}}$. More precisely, following Feferman, we will employ a schematic language $\mathcal{L}^{\rightarrow}_{\mathtt{Tr}}(P)$ (and sub-languages theoreof) featuring a fresh schematic predicate symbol P, which is assumed to behave classically.

Definition 5. *The system* $\mathbf{KFL}^*$ *in* $\mathcal{L}^{\rightarrow}_{\mathtt{Tr}}(P)$ *extends* **KFL** *with*

(i) $\forall x(P(x) \vee \neg P(x))$;

(ii) *Disquotational axioms for* P:

(**KFL**P1) $\qquad \mathtt{Tr}(\ulcorner P\dot{x}\urcorner) \Rightarrow P(x)$

(**KFL**P2) $\qquad P(x) \Rightarrow \mathtt{Tr}(\ulcorner P\dot{x}\urcorner)$

(iii) *The substitution rule:*

$$\frac{\Gamma(P) \Rightarrow \Delta(P)}{\Gamma(A/P) \Rightarrow \Delta(A/P)} \text{ for } A \text{ in } \mathcal{L}^{\rightarrow}_{\mathtt{Tr}}(P); \Gamma, \Delta \subseteq \mathcal{L}^{\rightarrow}_{\mathbb{N}}(P).$$

The semantics given in §4.1 can be modified to provide a class of fixed-point models for $\mathbf{KFL}^*$. We call Φ_X the result of relativizing the operator from §4.1 to an arbitrary $X \subseteq \omega$.[8] In particular, this means supplementing the positive inductive definition associated with Φ with the clause:

> a sentence Pz, with $\mathtt{Cterm}_{\mathcal{L}_{\mathtt{Tr}}}(z)$, is in the extension of the truth predicate (relativized to X) iff $\mathtt{val}(z) \in X$.

This modification clearly does not compromise the monotonicity of the operator. Therefore, let MIN_{Φ_X} be the minimal fixed point of Φ_X, and MAX_{Φ_X} its maximal one. For any X, we then obtain the minimal **HYPE** model

$$\mathfrak{M}^{\mathtt{min}}_{\Phi_X} := (\{\mathrm{MIN}_{\Phi_X}, \mathrm{MAX}_{\Phi_X}\}, \subseteq, *)$$

[8]Feferman [Fef91] provides a relativized fixed-point construction to arbitrary subsets of natural numbers and establishes the soundness of $\mathbf{KF}^*$.

In $\mathfrak{M}^{\mathtt{min}}_{\Phi_X}$, all arithmetical vocabulary is interpreted standardly at the two states. Only the interpretation of the truth predicate varies. Our notation reflects this.

Proposition 3. *If* $\mathbf{KFL}^* \vdash \Gamma \Rightarrow \Delta$*, then for all* X*,* $\mathfrak{M}^{\mathtt{min}}_{\Phi_X} \Vdash \Gamma \Rightarrow \Delta$*.*

Proof. By induction on the length of the derivation in $\mathbf{KFL}^*$.

We consider the case of the substitution rule applied to an arithmetical sequent $\Gamma(P) \Rightarrow \Delta(P)$. That is, our proof ends with

$$\frac{\Gamma(P) \Rightarrow \Delta(P)}{\Gamma(B/P) \Rightarrow \Delta(B/P)}$$

with $B(x)$ an arbitrary formula of $\mathcal{L}^{\rightarrow}_{\mathtt{Tr}}$.

By induction hypothesis, for all X, $\mathfrak{M}^{\mathtt{min}}_{\Phi_X} \Vdash \Gamma(P) \Rightarrow \Delta(P)$. Since $\Gamma(P) \Rightarrow \Delta(P)$ is arithmetical, for all interpretations Y of P, $(\mathbb{N}, Y) \models \Gamma(P) \Rightarrow \Delta(P)$. Then, following [Fef91], we can let:

$$Z = \{n \mid \mathfrak{M}^{\mathtt{min}}_{\Phi_X} \Vdash \Gamma(B(\overline{n})/P) \Rightarrow \Delta(B(\overline{n})/P)\}$$

to obtain that:

$$\mathfrak{M}^{\mathtt{min}}_{\Phi_X} \Vdash \Gamma(B/P) \Rightarrow \Delta(B/P).$$

□

5.2. **Proof-theoretic analysis.** We first consider the proof-theoretic lower-bound for $\mathbf{KFL}^*$. We adapt to the present setting the strategy outlined in [FS00, p. 84]. In particular, they formalize a (relativized) A-Jump hierarchy, which is a hierarchy of sets obtained by iterating an arithmetical operator expressed by an arithmetical formula $A(X, \theta, y)$. In particular, this hierarchy is relativized to a set of natural numbers expressed by a predicate P. For our purposes, it's useful to formalize such a hierarchy by replacing membership in second-order parameters with suitable truth ascriptions. In order to achieve this, we employ Feferman's strategy in [Fef91] in which the stages of the Turing jump-hierarchy are represented by means of suitable primitive recursive functions on codes of $\mathcal{L}_{\mathtt{Tr}}$-formulas. Specifically, we encode in suitable PR functions the stages of the A-Jump hierarchy. We denote with $A(\mathtt{Tr}x(\dot{u}), \eta, y)$ the result of replacing every occurrence of $u \in X$ with $\mathtt{Tr}\,\mathrm{sub}(x, \mathrm{num}(u))$ in $A(X, \eta, y)$. We let:

$$\begin{aligned}
f_0 &:= \ulcorner P\overline{n} \urcorner \\
f^a(n) &:= \ulcorner \exists q(n_0 = q \wedge q \prec a \wedge \mathtt{Tr} f_{\dot{q}}\, n_1) \urcorner \\
f_a(n) &:= \ulcorner A(\mathtt{Tr} f^a, a, n) \urcorner \\
Y_a(n) &:= \mathtt{Tr} f_a(\overline{n})
\end{aligned}$$

An essential ingredient of the lower-bound proof for $\mathbf{KFL}^*$ is the 'disquotational' behaviour of our truth predicate for stages in the hierarchy that are provably well-founded.

Lemma 12. *If we have* $\mathtt{TI}_\alpha(\mathcal{L}_{\mathtt{Tr}})$*, then for all* b*, with* $0 \prec b \prec \alpha$

$$Y_b(n) \leftrightarrow A(\mathtt{Tr} f^b, b, n).$$

Proof. For all a and all n, we can show that $\mathrm{Sent}_a(f^a(\overline{n}))$ and $\mathrm{Sent}_a(f_a(\overline{n})))$ by transfinite induction on a making use of the properties of the ramified truth predicates such as, for $\lambda \prec \alpha$ limit:

$$\forall b \prec \lambda\big(\mathtt{Tr}_\lambda(\mathtt{Tr}_b t) \leftrightarrow \mathtt{Tr}_b \mathrm{val}(t)\big).$$

□

Another key component in the proof of the lower-bound for predicative system is Schütte's Veblen-jump formula [Sch77, p. 185]. One first defines the functions:

$$\begin{aligned} &\mathtt{e}(0) = 0 && \mathtt{h}(0) = 0 \\ &\mathtt{e}(\omega^\eta) = \eta && \mathtt{h}(\omega^\eta) = 0 \\ &\mathtt{e}(\omega^{\eta_1} + \ldots + \omega^{\eta_n}) = \eta_n && \mathtt{h}(\omega^{\eta_1} + \ldots + \omega^{\eta_n}) = \omega^{\eta_1} + \ldots + \omega^{\eta_{n-1}} \end{aligned}$$

with $\eta_n \preceq \ldots \preceq \eta_1$. Recall that the Gentzen jump formula – with $\to$ the **HYPE** conditional – is given by:

$$\mathcal{J}(B, \xi) := \forall \eta(\forall \zeta \prec \eta\, B(\zeta) \to \forall \zeta \prec \eta + \xi\, B(\zeta))$$

The more complex Veblen jump formula $\mathcal{A}$ on which we build our $\mathcal{A}$-*hierarchy*:

$$\mathcal{A}(Y, \xi, y) := \forall \zeta(h(\xi) \preccurlyeq \zeta \prec \xi\, \mathcal{J}(Y_\zeta, \varphi(e(\xi), y)))$$

We can then show an equivalent of Schütte's Lemma 9 in [Sch77, p. 186], establishing the progressiveness of the stages of the $\mathcal{A}$-hierarchy:

Lemma 13. *If* $\mathtt{TI}(\alpha, \mathcal{L}_{\mathtt{Tr}}^{\to})$ *is provable in* $\mathbf{KFL}^*$ *for* $\alpha < \Gamma_0$, *then* $\mathbf{KFL}^*$ *proves:*

$$\forall \zeta(0 \prec \zeta \prec \alpha \wedge (\forall \theta \prec \zeta\, \mathrm{Prog}(Y_\theta^{\mathcal{A}}) \to \mathrm{Prog}(Y_\zeta^{\mathcal{A}}))).$$

Proof. One first shows that the following claims

(19) $\zeta \prec t$

(20) $\forall x \prec \zeta\, \mathrm{Prog}(Y_x^{\mathcal{A}})$

(21) $\forall y \prec \eta\, Y_\zeta^{\mathcal{A}}(y)$

(22) $\forall y(l(y) < l(\theta) \to (y \prec \varphi_{e(\zeta)}(\eta) \to (\forall z(h(\zeta) \preccurlyeq z \prec \zeta \to \mathcal{J}(Y_z^{\mathcal{A}}, y))))$

(23) $\theta \prec \varphi_e(a)(\zeta)$

(24) $h(\zeta) \preceq \xi \prec \zeta$

entail $\mathcal{J}(Y_\xi^{\mathcal{A}}, \theta)$. By the principle of induction:

(25) $$\forall x(\forall y(l(y) < l(x) \to \phi(y)) \to \phi(x)) \to \phi(t)$$

applied to the formula

$$\phi(y) :\leftrightarrow y \prec \varphi_{e(\zeta)}\eta \to \forall z(h(\zeta) \preccurlyeq z \prec \zeta\, \mathcal{J}(Y_z^{\mathcal{A}}, y)),$$

we obtain

$$\forall x \prec \zeta\, \texttt{Prog}(Y^{\mathcal{A}}_x) \to (\forall y \prec \eta\, Y^{\mathcal{A}}_\zeta(y) \to (\theta \preceq \varphi_{e(\zeta)}\eta \to \forall z(h(\zeta) \preccurlyeq z \prec \zeta\ \mathcal{J}(Y^{\mathcal{A}}_z, \theta)))). \tag{26}$$

Using Lemma 6 and the properties of progressiveness we obtain:

$$\texttt{Prog}(Y^{\mathcal{A}}_\xi) \to (\forall x \prec \varphi_{e(\zeta)}(\eta)\mathcal{J}(Y^{\mathcal{A}}_\xi, x) \to \mathcal{J}(Y^{\mathcal{A}}_\xi, \varphi_{e(\zeta)}\eta) \tag{27}$$

Therefore,

$$\forall x \prec \zeta\, \texttt{Prog}(Y^{\mathcal{A}}_x) \to \forall y \prec \eta\, Y^{\mathcal{A}}_\zeta(y) \to \forall z(h(\zeta) \preccurlyeq z \prec \zeta\ \mathcal{J}(Y^{\mathcal{A}}_z, \varphi_{e(\zeta)}\eta)), \tag{28}$$

However, by the definition of $\mathcal{A}$,

$$\forall x \prec \zeta\, \texttt{Prog}(Y^{\mathcal{A}}_x) \to \forall y \prec \eta\, Y^{\mathcal{A}}_\zeta(y) \to \mathcal{A}(Y^{\mathcal{A}}, \zeta, \eta). \tag{29}$$

Therefore, by Lemma 12,

$$\forall x \prec \zeta\, \texttt{Prog}(Y^{\mathcal{A}}_x) \to \forall y \prec \eta\, Y^{\mathcal{A}}_\zeta(y) \to Y^{\mathcal{A}}_\zeta(\eta). \tag{30}$$

□

Proposition 4. *If* $\texttt{TI}_{\gamma_n}(P)$ *is derivable in* $\mathbf{KFL}^*$*, then* $\texttt{TI}_{\varphi_{\gamma_n}0}(P)$ *is derivable in* $\mathbf{KFL}^*$*.*

Proof. We assume $\texttt{Prog}(P)$. If $\texttt{TI}_{\gamma_n}(P)$ is derivable in $\mathbf{KFL}^*$, then by Corollary 3 and the determinateness of P we can show $\texttt{TI}_{\omega^{\gamma_n}+1}(P)$. By the substitution rule we get that the hierarchy predicates are well-defined. Additionally we can prove that

$$\forall \zeta \prec \omega^{\gamma_n} + 1\ \forall x(Y_\zeta(x) \vee \neg Y_\zeta(x)) \tag{31}$$

Notice that by (31) we can reformulate this fragment of the hierarchy by replacing all the occurrences of the HYPE-conditional by the material conditional. Therefore, we have:

$$\forall \zeta \prec \omega^{\gamma_n} + 1(Y_\zeta(x) \leftrightarrow \mathcal{A}(f^\zeta, \zeta, x)) \tag{32}$$

By the previous lemma, for $a \prec \omega^{\gamma_n} + 1$,

$$\forall b \prec a\ \texttt{Prog}(Y_b) \to \texttt{Prog}(Y_a) \tag{33}$$

and therefore, by the substitution rule applied to $\texttt{TI}_{\gamma_n}(P)$ and (33), that $\texttt{Prog}(Y_{\omega^{\gamma_n}})$, which entails $Y_{\omega^{\gamma_n}}(\overline{0})$.

Using (32), we have:

$$\forall \zeta(h(\omega^{\gamma_n}) \preceq \zeta \prec \omega^{\gamma_n}\ \mathcal{J}(Y_\zeta, \varphi_{e(\omega^{\gamma_n})}0) \tag{34}$$

However, since $\texttt{h}(\omega^{\gamma_n}) = 0$ and $\texttt{e}(\omega^{\gamma_n}) = \gamma_n$ we can then infer

$$\forall \zeta \prec \omega^{\gamma_n}(\forall y(\forall x \prec y Y_\zeta(x) \to \forall x \prec y + \varphi_{\gamma_n}0\ Y_\zeta(x))) \tag{35}$$

By letting $\zeta = y = 0$, we obtain $\forall x \prec \varphi_{\texttt{e}(\omega^{\gamma_n})}0\ P(x)$, as desired. □

Corollary 7. $\mathbf{KFL}^*$ *defines the truth predicates of* $\mathbf{RT}_{<\alpha}$*, for* $\alpha \prec \Gamma_0$*.*

Following the characterization of predicative analysis in terms of ramified systems given in [Fef64, Fef91], and the relationships between ramified truth and ramified analysis studied there, one can then conclude that the systems of ramified analysis below Γ_0 are proof-theoretically reducible to our system $\mathbf{KFL}^*$.

The argument employed in the previous section to show that **KFL** can be proof-theoretically reduced – w.r.t. arithmetical sentences – to **KF** can be lifted to $\mathbf{KFL}^*$. One can consider the system $\mathbf{KF}^* - \mathtt{Ref}^*(\mathtt{PA}(P))$ in [Fef91] –, and slightly modify the translations σ, τ from Definition 4: in particular, we let

$$(Ps)^\sigma = (Ps)^\tau = Ps \qquad\qquad (\neg Ps)^\tau = \neg Ps.$$

Then, by induction on the length of proof in $\mathbf{KFL}^*$, we can prove:

Proposition 5. *If* $\mathbf{KFL}^* \vdash \Gamma \Rightarrow \Delta$*, then* $\mathbf{KF}^* \vdash (\bigwedge \Gamma \to \bigvee \Delta)^\sigma$.

Given the analysis of $\mathbf{KF}^*$ given in [Fef91], the combination of Propositions 5 and 4 yields a sharp proof-theoretic analysis also for $\mathbf{KFL}^*$:

Corollary 8. $|\mathbf{KFL}^*| = |\mathbf{KF}^*| = \Gamma_0$.

References

[Can89] Andrea Cantini. Notes on formal theories of truth. *Zeitschrift für mathematische Logik und Grundlagen der Mathematik*, 35:97–130, 1989.

[Fef64] Solomon Feferman. Systems of predicative analysis. *The Journal of Symbolic Logic*, 29(1):1–30, 1964.

[Fef84] Solomon Feferman. Toward useful type-free theories.i. *The Journal of Symbolic Logic*, 49(1):75–111, 1984.

[Fef91] Solomon Feferman. Reflecting on incompleteness. *The Journal of Symbolic Logic*, 56:1–47, 1991.

[FHN17] Martin Fischer, Leon Horsten, and Carlo Nicolai. Iterated reflection over full disquotational truth. *Journal of Logic and Computation*, 27(8):2631–2651, 2017.

[Fie08] H. Field. *Saving Truth from Paradox.* Oxford University Press, 2008.

[Fie20] H. Field. The power of naïve truth. *The Review of Symbolic Logic*, 2020. Forthcoming.

[Fis20] Martin Fischer. A sequent system for hype. draft, 2020.

[FS00] Solomon Feferman and T. Strahm. The unfolding of non-finitist arithmetic. *Annals of Pure and Applied Logic*, 104:75–96, 2000.

[Gör71] Sabine Görnemann. A logic stronger than intuitionism. *The Journal of Symbolic Logic*, 36(2):249–261, 1971.

[GSS09] D. M. Gabbay, V. B. Shehtman, and D. P. Skvortsov. *Quantification in nonclassical logic. Vol. 1*, volume 153 of *Studies in Logic and the Foundations of Mathematics*. Elsevier B. V., Amsterdam, 2009.

[Hal14] Volker Halbach. *Axiomatic Theories of Truth*. Cambridge University Press, Cambridge, UK, revised edition, 2014.

[HH06] Volker Halbach and Leon Horsten. Axiomatizing Kripke's theory of truth. *The Journal of Symbolic Logic*, 71:677–712, 2006.

[HP93] Petr Hájek and Pavel Pudlák. *Metamathematics of First-Order Arithmetic*. Springer Verlag, Berlin, 1993.

[Kri75] Saul Kripke. Outline of a theory of truth. *The Journal of Philosophy*, 72:690–716, 1975.

[LE83] E.G.K. López-Escobar. A second paper "on the interpolation theorem for the logic of constant domains". *The Journal of Symbolic Logic*, pages 595–599, 1983.

[Lei19] Hannes Leitgeb. Hype: A system of hyperintensional logic (with an application to semantic paradoxes). *Journal of Philosophical Logic*, 48:305–405, 2019.

[Nic17] Carlo Nicolai. Provably true sentences across axiomatizations of Kripke's theory of truth. *Studia Logica*, 106(1):101–130, 2017.

[NP01] Sara Negri and Jan von Plato. *Structural Proof Theory*. Cambridge University Press, 2001.

[Poh09] W. Pohlers. *Proof Theory, The first step into impredicativity*. Springer Verlag, 2009.

[Sch77] Kurt Schütte. *Proof Theory*. Springer Verlag, Berlin, 1977.

[Spe20] Stanislav O. Speranski. *Negation as a modality in a quantified setting*. 2020. draft.

[Tak87] Gaisi Takeuti. *Proof Theory*. North Holland, Amsterdam, second edition, 1987.

[TS00] A.S. Troelstra and H. Schwichtenberg. *Basic Proof Theory*. Cambridge University Press, 2 edition, 2000.

[Vis84] Albert Visser. Semantics and the liar paradox. In *Handbook of Philosophical Logic*, pages 149–240. Gabbay, D., 2 edition, 1984.

[Woo84] Peter W. Woodruff. Paradox, truth and logic part i: Paradox and truth. *Journal of Philosophical Logic*, 13(2):213–232, 1984.

M.Fischer@lrz.uni-muenchen.de, LMU Munich.

carlo.nicolai@kcl.ac.uk, King's College London.

pablodopicofernndez@yahoo.es, King's College London.